Sampling and Multirate Techniques for Complex and Bandpass Signals

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Topics:

• Complex signals and systems
• Sampling of complex signals
• Frequency translations using mixing and multirate operations
• Real and I/Q sampling of bandpass signals
• Nonidealities in sampling and A/D-conversion
• Delta-sigma conversion principle
Complex Signals and Systems

In telecommunications signal processing, it is common to use the notion of complex signals.

Continuous- and discrete-time complex signals are denoted here as

\[ x(t) = x_R(t) + jx_I(t) \quad x(k) = x_R(k) + jx_I(k) \]

In practical implementations, complex signals are nothing more than two separate real signals carrying the real and imaginary parts.

A complex linear time-invariant system is represented by two real impulse responses

\[ h(k) = h_R(k) + jh_I(k) \]

or the corresponding two real-coefficient transfer functions

\[ H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) \]

In the general case, to implement a complex filter for a complex signal, four separate real filters need to be implemented

\[ y(k) = x(k)*h(k) = (x_R(k) + jx_I(k))*(h_R(k) + jh_I(k)) \]
\[ = x_R(k)*h_R(k) - x_I(k)*h_I(k) + j(x_R(k)*h_I(k) + x_I(k)*h_R(k)) \]
Important Special Cases of Complex Signals

Real signal spectrum, the corresponding analytic signal spectrum, and the analytic signal spectrum translated to be centered at 0 frequency:

Real bandpass signal and the corresponding analytic bandpass signal:
Important Special Cases of Complex Signals

Other side of the desired signal spectrum suppressed by a practical phase splitter (e.g., for improving image attenuation in case of I/Q downconversion):
Sampling Theorem

The sampling theorem says that a (real or complex) lowpass signal limited to the frequency band \([-W, W]\) can represented completely by discrete-time samples if the sampling rate \((1/T)\) is at least \(2W\).

In case of a complex signal, each sample is, of course, a complex number.

In general, discrete-time signals have periodic spectra, where the continuous-time spectrum is repeated around frequencies \(\pm 1/T, \pm 2/T, \pm 3/T, \ldots\)

In case of complex signals, it is not required that the original signal is located symmetrically around 0 if no overlapping occurs in the frequency domain.

Any part of the periodic signal can be considered as the useful part. This allows many possibilities for multirate processing of bandpass signals.

In general, the key criterion is that no destructive aliasing effect occur.
Real vs. Complex Discrete-Time Signals

Real signal:

Here $2W$ real samples per second are sufficient to represent the signal.

Complex signal:

Here $W$ complex samples per second are sufficient.

- The resulting rates of real-valued samples are the same.
- However, the quantization effects may be quite different. (Recall from the standard treatment of SSB that Hilbert-transformed signals may be difficult.)
Frequency Translation

One key operation is the frequency translation of a signal spectrum from one center frequency to another.

Conversions between baseband and bandpass representations (modulation and demodulation) are special cases of this.

We consider two different ways to do the frequency translation: mixing and multirate operations, i.e., decimation and interpolation.

In case of multirate operations, we assume for simplicity that the following two sampling rates are used:

- low sampling rate: \( \frac{f_s}{N} = \frac{1}{NT} \)
- high sampling rate: \( f_s = \frac{1}{T} \)
Mixing for Complex Discrete-Time Signals

\[ y(k) = e^{j2\pi f_{LO} k T} x(k) = e^{j\omega_{LO} k} x(k) \]

This produces a pure frequency translation of the spectrum by \( f_{LO} \).

Important special cases are:

1. \( f_{LO} = f_s / 2 = 1 / 2T \)
   
in which case the multiplying sequence is +1, -1, +1, -1, ...
   
   This case can be applied to a real signal without producing a complex result. Converts a lowpass signal to a highpass signal, and vice versa.

2. \( f_{LO} = f_s / 4 = 1 / 4T \)
   
in which case the multiplying sequence is +1, j, -1, -j, +1, j, ...

Complex Bandpass Filters

Certain types of complex filters based on Hilbert transformers can be design using standard filter design packages, like Parks-McClellan routine for FIR filters.

Another way to get complex bandpass filters is through frequency translations:

**Real prototype filter:**

**Complex bandpass filter:**

Transformation for frequency response and transfer function:

\[
H(e^{j\omega}) \rightarrow H(e^{j(\omega - \omega_c)}) \\
H(z) \rightarrow H(ze^{-j2\pi f_c T})
\]

Transformation for block diagram:

If 1/T is an integer multiple of \(f_c\), this might be much easier than in the general case, see the special cases of the previous page.
Example of a Complex Bandpass Filters: Frequency Translated FIR

Frequency translation by $f_s/4 \Rightarrow$ Analytic bandpass filter with passband around $f_s/4$.

Impulse response translated as:

$$h_0, h_1, h_2, h_3, h_4, \ldots, h_N \downarrow h_0, jh_1, -h_2, -jh_3, h_4, \ldots, (j)^N h_N$$
FIR Filter with Frequency Translation by $f_s/4$

(i) Real input signal

If the filter length is odd and if it is a linear-phase design, the coefficient symmetry can be exploited.

(ii) Complex input signal

The possible coefficient symmetry can always be exploited.
Interpolation for Complex Signal

Sampling rate increase produces a periodic spectrum, and the desired part of the spectrum is then separated by an (analytic) bandpass filter.

\[
\text{COMPLEX BP-FILTER RESPONSE}
\]

\[
\frac{n}{NT} - \frac{1}{T} \quad 0 \quad \frac{n}{NT}
\]

\[
\frac{1}{NT} \quad 0 \quad \frac{1}{NT}
\]

\[
\text{a)}
\]

\[
\text{b)}
\]
Decimation for Complex Signal

Sampling rate decrease produces aliasing, such that the original band is at one of the image bands of the resulting final band.

The signal has to be band-limited to a bandwidth of $\frac{1}{NT}$ before this operation can be done.
Combined Multirate Operations for Complex Signal

Combining decimation and interpolation, a frequency shift by \( n / NT \) can be realized, where \( n \) is an arbitrary integer.

\[
\begin{align*}
\text{It can be seen that the low sampling rate, limited to be higher than the signal bandwidth, determines the resolution of the frequency translations based on multirate operations.}
\end{align*}
\]

If, for example, a bandpass signal is desired to be translated to the baseband form, this can be done using multirate operations if and only if the carrier frequency is a multiple of the low sampling rate.
Combining Mixing and Multirate Operations for Complex Signals

A general frequency shift of $f_O = \frac{n}{NT} + f_\Delta$ can be done in the following two ways:

(1) Direct frequency conversion by $f_O$ using mixing.

(2) Conversion using multirate operations by $\frac{n}{NT}$ followed by a mixing with $f_\Delta$ (or vice versa).

The differences in these two approaches are due to the possible filtering operations associated with the multirate operations, and aliasing/reconstruction filters in case of mixed continuous-time/discrete-time processing.

Assuming ideal filtering, these two ways would be equivalent.
Example of Combining Mixing and Multirate Operations

Conversion from bandpass to baseband representation and decimation to symbol rate, i.e., I/Q-demodulation.

Assume that
- $N=6$, $f_0=4/(6T)+f_\Delta$.
- The required complex bandpass filter is obtained from an FIR filter of length 50 by frequency translation.

The following three ways are equivalent but lead to different computational requirements (the required real multiplication rates at input rate are shown; not exploiting possible coefficient symmetry):

<table>
<thead>
<tr>
<th>Case (i)</th>
<th>Case (ii)</th>
<th>Case (iii)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Filter</td>
<td>100</td>
<td>100/6</td>
</tr>
<tr>
<td>Mixer</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>104</td>
<td>18.7</td>
</tr>
</tbody>
</table>
Example of Combining Mixing and Multirate Operations (continued)

Notes:

(i) Complex bandpass filter, real inputs
    => 100 real multipliers needed for filter

(ii) Real lowpass filter, complex input to filter
    => 100 real multipliers needed for filter

    - Decimation can be combined efficiently with the filter. Utilizing coefficient symmetry is easiest in this case.

(iii) As (i) but decimation can be included efficiently with the filter.

    - Mixing and LO generation done at lower rate and thus easier to implement.

Here we have not taken use of the possible coefficient symmetry, which may reduce the multiplication rates by 1/2 in all cases.

In general, mixing is a memoryless operation, so up-sampling and down-sampling operations can be commuted with it in block diagram manipulations.
Frequency Translation for Real Signals

Mixing and multirate operations can be done in similar way for real signals. The difference is that the two parts of the spectrum, on the positive and negative frequency axis, and their images, must be accommodated in the spectrum.

(1) Mixing

Mixing produces two translated spectral components. The image band appearing on top of the desired band after mixing must be suppressed before mixing.

\[
\begin{align*}
&-f_c - f_{LO} \quad -f_c + f_{LO} \quad 0 \quad f_c - f_{LO} \quad f_c + f_{LO} \\
&\cos(\omega_{LO}t)
\end{align*}
\]

(2) Multirate operations

In case of decimation, to avoid destructive aliasing effects, the signal to be translated must be within one of the intervals

\[
\left\{ \frac{n}{NT}, \quad \frac{n}{NT} + \frac{1}{2NT} \right\} \quad \text{or} \quad \left\{ \frac{n}{NT} - \frac{1}{2NT}, \quad \frac{n}{NT} \right\}
\]

Otherwise destructive aliasing will occur. In the latter case, the spectrum will be inverted.
Interpolation for Real Bandpass Signal

\[ x(n) \xrightarrow{\uparrow L} w(m) \xrightarrow{\text{Filter}} y(m) \]

\[ X(f) \]

\[ W(f) \]

\[ Y(f)|_{k=2} \]

\[ Y(f)|_{k=3} \]
Decimation for Real Bandpass Signal

\[ x(n) \xrightarrow{\mathbb{R}} x_{BP}(n) \quad f_s \xrightarrow{1/L} y(m) \quad f_s/L \]

\[ X_{BP}(f) \]

\[ Y(f) \]

\[ X_{BP}(f) \]

\[ Y(f) \]
Example of Down-Conversion: I/Q-Demodulation

It is usually a good idea to keep the signal as a real signal as long as possible, because after converting to complex form, all subsequent signal processing operations require double computational capacity compared to the corresponding real algorithms.
Real Bandpass Sampling

Down-conversion can also be implemented by sampling a bandpass signal. Any part of the periodic spectrum can be selected for further processing.

Concerning the sampling frequency, it is sufficient that no aliasing appears on top of the desired band.

In general, the feasible sampling frequencies are determined from $W$, $B$ (useful signal bandwidth), and $f_s$. Minimum sampling frequency is $B+W$, which is adequate in the case where the center frequency of the desired signal is $f_s/4+kf_s$: 

$$W+B=f_s$$
Quadrature Sampling

In this case we are sampling the complex analytic signal obtained by a phase-splitter:

Of course, in practise the image bands in between the shown periodic replicas are not completely attenuated but only suppressed to a level that is determined by the amplitude and phase imbalances in the phase splitter & sampler & ADC blocks.

The gain and phase imbalance analysis of quadrature down-conversion applies also to this case.
Second-Order Sampling

Quadrature sampling can be approximated by the following structure:

At the carrier frequency, the sampling time offset corresponds exactly to the $90^\circ$ phase shift. Farther away from the center frequency this is only approximative, but for relatively narrowband signals, it works. The nonideality can be evaluated using the phase imbalance analysis.
Analysis of Second-Order Sampling

This system works perfectly at the carrier frequency but only approximately at other frequencies. At frequency $f_c + f_\Delta$, a time-shift of $1/4f_c$ corresponds to a phase shift of

$$\frac{1}{4f_c} \cdot 2\pi = \frac{1 + \frac{f_\Delta}{f_c}}{2} \pi \text{ rads}$$

We are actually dealing with phase imbalance and the image rejection formula for quadrature mixing can be utilized (see slide 83080RA/16). The resulting image rejection is:

$$R = \frac{1 - \cos \phi}{1 + \cos \phi} = \frac{1 - \cos \left( \frac{f_\Delta}{f_c} \cdot \frac{\pi}{2} \right)}{1 + \cos \left( \frac{f_\Delta}{f_c} \cdot \frac{\pi}{2} \right)}$$

**Example:** $f_c = 1 \text{ GHz}$

<table>
<thead>
<tr>
<th>$f_\Delta$</th>
<th>$f_\Delta/f_c$</th>
<th>Phase imbalance</th>
<th>Image rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 MHz</td>
<td>0.0001</td>
<td>0.009°</td>
<td>82.1 dB</td>
</tr>
<tr>
<td>1 MHz</td>
<td>0.001</td>
<td>0.09°</td>
<td>62.1 dB</td>
</tr>
<tr>
<td>10 MHz</td>
<td>0.01</td>
<td>0.9°</td>
<td>42.1 dB</td>
</tr>
<tr>
<td>100 MHz</td>
<td>0.1</td>
<td>9°</td>
<td>22.1 dB</td>
</tr>
</tbody>
</table>
Problems with Wideband Sampling

Analog to Digital Converter (ADC)

Sampling a wideband signal, containing several channels is a tempting approach for designing a flexible radio receiver. However, there are some great challenges to do this. The strongest signal in the ADC input signal band should be in the linear range of the ADC. When the desired signal is weak, a large ADC dynamic range is needed, the resolution of the converter has to be many bits, e.g., 14 ... 17 bits.

![Diagram showing magnitude and required dynamic range]

**Sampling**

The sampling to get a discrete time signal is done usually with a track-and-hold circuit (T/H).

In practical sampling clocks and sampling circuits, there are unavoidable random variations in the sampling instants, *sampling aperture jitter*. In bandpass sampling, the requirements for aperture jitter become very hard.
Quantization Noise in ADCs

In general, the maximum S/N-ratio for an A/D-converter is estimated by

\[ SNR = 6.02n + 1.76 + 10\log_{10}\left[\frac{f_s}{2B}\right] \] (dB)

where \( n \) is the number of bits,
\( B \) is the useful signal bandwidth,
\( f_s \) is the sampling rate.

The last term takes into account the processing gain due to oversampling in relation to the useful signal band. When the quantization noise outside that useful signal band is filtered away, the overall quantization noise power is reduced by the factor \( f_s/2B \).

The number of additional bits needed to quantize a wideband signal can be estimated by:

\[ 10\log_{10}\left[\frac{P_B}{P_d}\right]/6 \] bits

where \( P_B \) is the worst case power in the full band,
\( P_d \) is the minimum useful signal power.

Usually, in radio communications receivers, the worst case power is determined from the adjacent channel or blocking signal specifications.
Spurious-Free Dynamic Range

Practical ADC's have also discrete spectral frequency components, spurious signals (or spurs), in addition to the flat quantization noise.

In many applications, the spurious-free dynamic range, SFDR, is the primary measure of the dynamic range of the converter.
Track&Hold Circuit Nonidealities

Advanced bandpass sampling approaches could mean that we are sampling a tens-of-MHz to GHz-range signal with a relatively low sampling rate.

Noise Aliasing

Wideband noise at the sampling circuitry will be aliased to the signal band. In case of bandpass sampling, aliasing increases with increasing subsampling \((f_c/f_s)\) factor.

Therefore, it is important to have a good noise figure for the track&hold circuit and/or to have sufficient amplification in the analog front-end.
Aperture Jitter

Aperture jitter is the variation in time of the exact sampling instant, that causes phase modulation and results in an additional noise component in the sampled signal.

Aperture jitter is caused both by the sampling clock and the sampling circuit.
SNR Due to Sampling Jitter

The noise produced by aperture jitter is usually modelled as white noise, which results in a signal-to-noise ratio of

\[ SNR_{aj} = 20 \log_{10} \left( \frac{1}{2\pi f_{\text{max}} T_{a}} \right) \]

where \( f_{\text{max}} \) is the maximum frequency in the sampler input and \( T_{a} \) is the rms value of the aperture jitter.

This model is derived for a sinusoidal input signal, but applied also more generally, because no other models exist.

In critical test cases of the wideband sampling receiver application, the blocking signal is often defined as a sinusoidal signal, and the model is expected to work reasonably well.

**Example case:**

![Graph showing ADC performance with SNR vs. frequency][1]
About A/D-Conversion for SW Radio

It is obvious that the requirements for the T/H-circuit and A/D-converter are the main bottlenecks for implementing receiver selectivity with DSP.

One promising A/D-converter technology in this context is the sigma-delta (ΣΔ) principle.

- This principle involves low-resolution, high-speed conversion in a noise-shaping configuration, together with decimating noise filtering.

- In case of lowpass and bandpass sampling with suitable fixed center frequency, this principle can be combined nicely with the selectivity filtering part of the receiver.

Noise filtering in basic ADC:

Noise filtering in sigma-delta converter:
**Sigma-Delta Modulator**

A Special quantization method
- Different transfer functions for signal and noise

\[ Y(z) = z^{-L}X(z) + \left(1 - z^{-1}\right)^L E(z) \]

- Attenuates noise from the desired signal band, thus in-band quantization noise is

\[ \sigma_e^2 = \int_{-B}^{B} S_Q(f)df \approx \frac{\Delta^2}{12} \frac{\pi^{2L}}{(2L+1)} \left(\frac{2B}{F_S}\right)^{(2L+1)} \]

Oversampling ratio has great effect
- Doubling the ratio \(F_s/2B\), noise is decreased by factor \(3(2L+1)\) in dB.
- Number of quantization bits can be reduced

Noise can be filtered out by digital filters.